CS 599 Rounding Techniques in Approximation Algorithms	**OCRS**
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Matroid Basics

A matroid $M = (N, \mathcal{F})$ consists of elements $e \in N$ and a set of independent sets $\mathcal{F} \subseteq 2^N$. It follows to properties:

1. Downward-closed: For all $I \subseteq J$ with $J \in \mathcal{F}$, then $I \in \mathcal{F}$.
2. Matroid exchange: For all $I, J \in I$, if $|J| > |I|$, then there exists $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{F}$.

The rank of a set S is the size of the largest Independent Set (IS) in S:

$$\text{rank}(S) := \max\{|I| \mid I \in \mathcal{F}, I \subseteq S\}.$$

The span of a set S is the largest set of elements that has the same rank as S:

$$\text{span}(S) = \{e \in N \mid \text{rank}(S + \{e\}) = \text{rank}(S)\}.$$

Note that:

- $\text{rank}(S) = \text{rank}(\text{span}(S))$
- for $S \in \mathbb{E}$, $\text{rank}(S) = |S|$
- $\text{rank}(S) \leq \text{rank}(S')$ for $S \subseteq S'$.

A basis S is independent and maximal: $|S| = \text{rank}(S) = \text{rank}(N)$.

The independence polytope is convex hull of the independent sets: $\mathcal{P}_{\mathcal{F}} = \{x \in R^M_{\geq 0} \mid \forall S \subseteq \mathcal{F} \sum_{i \in S} x_e \leq \text{rank}(S)\}$.

The base polytope is the convex hull of the bases: $\mathcal{P}_{\mathcal{B}} = \{x \in P_{\mathcal{F}} \mid x(N) = \text{rank}(N)\}$ where $x(S) := \sum_{x_e \in S} x_i$.

We define two operations on matroids:

- Restriction: $M|_{N'} = (N', \mathcal{F} \cap 2^{N'})$. Think like a subgraph.
- Contraction: $M/N' = (N \setminus N', \mathcal{F}')$ where $I' = \{S \subseteq N \setminus N' \mid \text{rank}(S \cup N') = |S| + \text{rank}(N')\}$. That is, the matroid with the elements N' removed, and the independent sets of the contraction must be independent itself and not spanned by N'. Essentially saving room for independent sets in N', and N' need not be independent.

Online Contention Resolution Schemes by Feldman, Svensson, Zenklusen [SODA '16]

CRS already existed for offline problems. This paper focuses on *online* selection problems, such as prophet inequalities, stochastic probing, and more, with feasibility constraints, such as matroids, knapsack, and matching. Today we'll focus on matroid feasibility constraints, and use prophet inequalities as a running example.

Traditionally in the prophet inequality problem: v_e arrives online with $v_e \sim F_e$. The goal is to approximation $\mathbb{E}[\max_{e \in I} v_e]$ for some $I \in \mathcal{F}$. Usual algorithms set T_e and accept if $v_e \geq T_e$ and

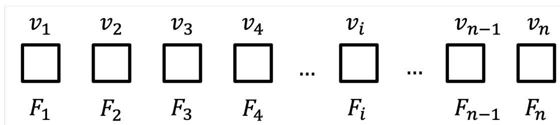


Figure 1: The prophet inequality problem.

feasible.

$$\begin{aligned}
\text{OPT} &= \sum_e \Pr[e \in \text{MWB}] \mathbb{E}[v_e \mid e \in \text{MWB}] \\
&\leq \sum_e x_e^* \, \mathbb{E}[v_e \mid v_e \geq t_e, \Pr[v_e \geq t_e] = x_e^*].
\end{aligned}$$

Converting the problem to this probability of being "on" or "active" and then just the expected weight is called the *ex ante relaxation*, as this is the LP where you solve for these probabilities.

For a matroid $M = (N, \mathcal{F})$, the independence polytope is

$$P_{\mathcal{F}} = \{x \in \mathbb{R}^N_{\geq 0} \mid \forall S \subseteq N \quad \sum_{e \in S} x_e \leq \text{rank}(S)\}.$$

This is the convex hull of independent sets.

- $R(x)$ is the set of *active elements*. That is, there is some distribution F_e over the weight z_e of element e, and for a threshold t_e, $e \in R(x)$ if $z_e \geq t_e$ which occurs with probability $x_e = \Pr[e \in R(x)]$.
- A *greedy OCRS* for $P_{\mathcal{F}}$, for any $x \in P_{\mathcal{F}}$, selects a downward-closed subfamily $\mathcal{F}_x \subseteq \mathcal{F}$. An element e is taken if it is active and if, with the other take elements so far, the combined set is feasible in $\mathcal{F}_x$.
- For $c \in [0, 1]$, a greedy OCRS for $P_{\mathcal{F}}$ is *c-selectable* if for any $x \in P_{\mathcal{F}}$,

  $$\Pr_{R(x)}[I \cup \{e\} \in \mathcal{F}_x \quad \forall I \subseteq R(x), I \in \mathcal{F}_x] \geq c \quad \forall e \in N,$$

  that is, every e is feasible to take with probability at least c, or, e is not spanned by any active independent set from I with probability at least c.

- For $b, c \in [0,1]$, a greedy OCRS for $P_{\mathcal{F}}$ is (b,c)-*selectable* if for any $x \in b \cdot P_{\mathcal{F}}$,

$$\Pr_{R(x)}[I \cup \{e\} \in \mathcal{F}_x \quad \forall I \subseteq R(x), I \in \mathcal{F}_x] \geq c \quad \forall e \in N.$$

Our goal in the next section is to construct a $(b, 1-b)$-selectable OCRS for matroids. This implies a $b(1-b)$-selectable OCRS for the non-scaled polytope. This is because we can "scale" each $x \in P_{\mathcal{F}}$ to $b \cdot P_{\mathcal{F}}$ by just considering each element e online independently with probability b before looking at its weight.

$$\begin{aligned}
\textsc{Alg} &= \sum_i \Pr[\text{over threshold}] \cdot \mathbb{E}[v_e \mid \text{ over } T_e] \cdot \Pr[\text{feasible to take}] \\
&\geq \sum_e x_e^* \, \mathbb{E}[v_e \mid v_e \geq T_e] \cdot b(1-b) \\
&\geq b(1-b) \cdot \textsc{opt}
\end{aligned}$$

Online, our algorithm: set $T_e = t_e$ unless it's infeasible to take e wrt $\mathcal{F}_x$, then $T_e = \infty$.

Constructing a $(b, 1-b)$-selectable greedy OCRS for matroids

Let $(M/S_1)|_{S_2}$ be the matroid of M with S_1 contracted and restricted to S_2. That is, I is independent in $(M/S_1)|_{S_2}$ if $I \subseteq S_2$ and $I \cup S_1'$ is an independent set in M for $\text{rank}(S_1') = \text{rank}(S_1)$.

We construct a chain of growing sets

$$\emptyset = N_\ell \subset N_{\ell-1} \subset \cdots \subset N_1 \subset N_0 = N$$

and our greedy OCRS accepts an active element $e \in N_i \setminus N_{i+1}$ if e together with the already accepted elements in $N_i \setminus N_{i+1}$ forms an independent set in $(M/N_{i+1})|_{N_i}$. That is,

$$\mathcal{F}_x = \{I \subseteq N \mid \forall I \cap (N_i \setminus N_{i+1}) \text{ is independent in } (M/N_{i+1})|_{N_i}\}.$$

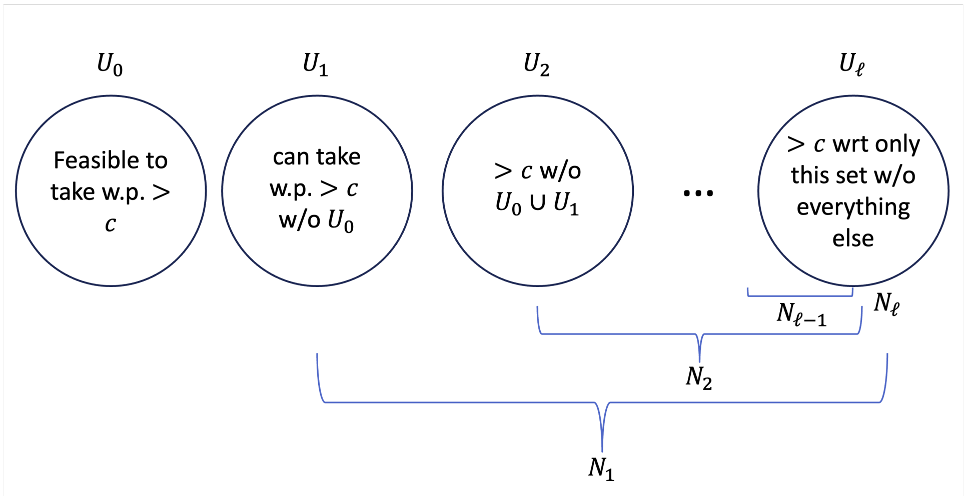
Observe that $\mathcal{F}_x$ is downward-closed and a sub-family of $\mathcal{F}$.

Intuitively, we will choose $N_{\ell-1}$ to be the elements that are very likely to be spanned by others, that is, it is very likely that we would take a different element that would preclude us from taking the elements of $N_{\ell-1}$. This way, we will save room for the elements in $N_{\ell-1}$ by requiring the feasible sets to form an independent set with a maximal rank set of $N_{\ell-1}$. By "saving room" for the elements in a hierarchical way, we will prioritize the sets such that each element has probability b of being selected. We do this by first separating all of the elements that have too low a probability of being chosen and then prioritize these, selecting subsequent subsets as necessary.

This shows how to construct the first subset N_1. Let $S_0 = \emptyset$ and $S_1 = \{e \in N \mid \Pr_{R(x)}[e \in \text{span}(R(x) \setminus \{e\})] > b\}$, the set of elements likely to be spanned by the active elements. Then

$$S_i = \{e \in N \mid \Pr_{R(x)}[e \in \text{span}(R(x) \cup S_{i-1}) \setminus \{e\}] > b\},$$

that is, S_i contains the elements that are likely to be spanned assuming that the elements of S_{i-1} are contracted (or equivalently appear with probability 1). Note that $S_{i-1} \subseteq S_i$ for all $i \geq 1$ and $S_i = S_{i-1}$ implies that $S_i = S_j$ for all $j > i$. Then define $N_1 = S = S_{|N|}$.



We now aim to show that we make progress with this construction, that is, $N_1 \subset N$. Since $S = N_1$ would only increase in size as coordinates of x increase, we show that we make progress for a maximal x, that is, $x \in b \cdot P_{\mathcal{B}}$ where $P_{\mathcal{B}} = \{x \in P_{\mathcal{F}} \mid x(N) = \text{rank}(N)\}$ is the base polytope of the matroid M, the set of all maximal vectors in $P_{\mathcal{F}}$. The following corollary (which uses the following lemma) shows that we do make progress.

Lemma 1. *It always holds that for $S \neq \emptyset$,*

$$\sum_{e \in N} x_e \text{Pr}_{R(x)}[e \in \text{span}(R(x) \cup S)] < b[x(N) + (1-b)\text{rank}(S)].$$

Proof. Let $S' = \{e_1, e_2, \dots, e_k\}$ be a basis of the matroid $M|_S$ obtained by first greedily selecting elements from S_1, then S_2, and so on.

Let A be a random set distributed like $R(x)$. For $j = 1, \dots, k$, if $e_j \notin \text{span}(A)$, add e_j to A. Call the distribution of the A to be μ.

That is, S is the full set of elements likely to spanned. S' is a greedily (inside to out) built basis of S. A is a realization of $R(x)$ with elements of S' that are not already spanned added. Then $\text{span}(S \cup R(x))$, $\text{span}(S' \cup R(x))$, and $\text{span}(A)$ are identically distributed, thus

$$\begin{aligned}
\sum_{e \in N} x_e \text{Pr}_{R(x)}[e \in \text{span}(R(x) \cup S)] &= \sum_{e \in N} x_e \text{Pr}_{A \sim \mu}[e \in \text{span}(A)] \\
&= \mathbb{E}_{A \sim \mu}[x(\text{span}(A))] \\
&\leq b \cdot \mathbb{E}_{A \sim \mu}[\text{rank}(\text{span}(A))] && x \in b \cdot P_{\mathcal{F}} \\
&= b \cdot \mathbb{E}_{A \sim \mu}[\text{rank}(A)] \\
&\leq b \cdot \mathbb{E}_{A \sim \mu}[|A|]
\end{aligned}$$

and

$$
\begin{aligned}
\mathbb{E}_{A\sim\mu}[|A|] &= x(N) + \sum_{j=1}^{k} \Pr[e_j \notin \operatorname{span}(R(x) \cup \{e_1, \ldots, e_{j-1}\})] && \text{def. of } A \\
&< x(N) + (1-b)k && \text{construction of } S \\
&= x(N) + (1-b) \cdot \operatorname{rank}(S)
\end{aligned}
$$

Hence

$$
\sum_{e \in N} x_e \Pr_{R(x)}[e \in \operatorname{span}(R(x) \cup S)] < b[x(N) + (1-b)\operatorname{rank}(S)].
$$

□

Corollary 1. *If $N \neq \emptyset$ then $N_1 = S \subset N$.*

Proof. If $S = \emptyset$, then the corollary is true. Otherwise,

$$
\begin{aligned}
x(S) &\le x(\operatorname{span}(R(x) \cup S)) && S \subseteq \operatorname{span}(R(x) \cup S) \\
&= \sum_{e \in N} x_e \Pr_{R(x)}[e \in \operatorname{span}(R(x) \cup S)] \\
&< b \cdot [x(N) + (1-b)\operatorname{rank}(S)] && \text{Lemma 1} \\
&\le b \cdot \operatorname{rank}(N) && \operatorname{rank}(S) \le \operatorname{rank}(N); x(N) = b \cdot \operatorname{rank}(N) \\
&= x(N) && x \in b \cdot P_{\mathcal{B}}
\end{aligned}
$$

Hence $x(S) < x(N)$, so $N \setminus S \neq \emptyset$. □

This gives the next theorem, and using their Theorem 1.9, the theorem after.

Theorem 2. *For $b \in [0,1]$, there exists a $(b, 1-b)$-selectable deterministic greedy OCRS for any matroid polytope $P_{\mathcal{F}} \subseteq [0,1]^N$ on ground set N.*

Theorem 3. *For $b \in [0,1]$, and k matroid polytopes $P_1, \ldots, P_k \subseteq [0,1]^N$ over a common ground set N, there exists a $(b, (1-b)^k)$-selectable deterministic greedy OCRS for $P = \bigcap_{i=1}^{k} P_i$.*